

STRONGLY σ -METRIZABLE SPACES ARE SUPER σ -METRIZABLE

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ABSTRACT. A topological space X is called *strongly σ -metrizable* if $X = \bigcup_{n \in \omega} X_n$ for an increasing sequence $(X_n)_{n \in \omega}$ of closed metrizable subspaces such that every convergence sequence in X is contained in some X_n . If, in addition, every compact subset of X is contained in some X_n , $n \in \omega$, then X is called *super σ -metrizable*. Answering a question of V.K. Maslyuchenko and O.I. Filipchuk, we prove that a topological space is strongly σ -metrizable if and only if it is super σ -metrizable.

Following [6] and [5], we define a topological space X to be

- *σ -metrizable* if X can be written as the union of an increasing sequence $(X_n)_{n \in \omega}$ of closed metrizable subspaces of X ;
- *strongly σ -metrizable* if X can be written as the union of an increasing sequence $(X_n)_{n \in \omega}$ of closed metrizable subspaces of X such that every convergent sequence in X is contained in some space X_n ;
- *super σ -metrizable* if X can be written as the union of an increasing sequence $(X_n)_{n \in \omega}$ of closed metrizable subspaces of X such that every compact subset of X is contained in some space X_n .

It turns out that the last two notions are equivalent (which answers a question of V.K. Maslyuchenko and O.I. Filipchuk). In the proof we shall use a recent result of Alas and Wilson [1] (see also [2]) on sequentially compact spaces. We recall that a topological space X is *sequentially compact* if each sequence in X has a convergent subsequence.

Theorem 1 (Alas, Wilson). *Each hereditarily Lindelöf compact space is sequentially compact.*

This theorem implies

Corollary 1. *Each compact σ -metrizable topological space X is hereditarily Lindelöf and sequentially compact.*

Proof. The space X , being σ -metrizable, admits a countable cover $\{X_n\}_{n \in \omega}$ by closed metrizable subspaces. Each metrizable subspace X_n , being closed in the compact space X , is compact and hence hereditarily Lindelöf. Then the union $X = \bigcup_{n \in \omega} X_n$ is hereditarily Lindelöf, too. By Theorem 1, the compact space X is sequentially compact. \square

For Hausdorff spaces the following characterization was proved in the PhD Thesis [4] of Filipchuk.

1991 *Mathematics Subject Classification.* 54E35, 54A20, 54D30.

Key words and phrases. strongly σ -metrizable space, super σ -metrizable space.

Theorem 2. *A topological space is strongly σ -metrizable if and only if it is super σ -metrizable.*

Proof. The “only if” part trivially follows from the observation that for any sequence $\{x_n\}_{n \in \omega} \subset X$, convergent to a point $x \in X$, the subspace $K = \{x\} \cup \{x_n\}_{n \in \omega}$ of X is compact.

To prove the “if” part, assume that a topological space X is strongly σ -metrizable. Then X is the union of an increasing sequence $(X_n)_{n \in \omega}$ of closed metrizable subspaces of X such that every convergent sequence in X is contained in some set X_n . To prove that X is super σ -metrizable, we need to show that any compact subset $K \subset X$ is contained in some X_n . To derive a contradiction, assume that for every $n \in \omega$ the complement $K \setminus X_n$ is not empty and hence contains some point x_n . By Corollary 1, the compact σ -metrizable space K is sequentially compact. Consequently, there exists an increasing number sequence $(n_i)_{i \in \omega}$ such that the subsequence $(x_{n_i})_{i \in \omega}$ of $(x_n)_{n \in \omega}$ converges in K . The choice of the sequence $(X_n)_{n \in \omega}$ guarantees that $\{x_{n_i}\}_{i \in \omega} \subset X_m$ for some $m \in \omega$. Choose a number $i \in \omega$ with $n_i \geq m$ and observe that the inclusion $x_{n_i} \in X_m$ contradicts the choice of $x_{n_i} \in K \setminus X_{n_i} \subset X \setminus X_m$. This contradiction shows that $K \subset X_n$ for some $n \in \omega$, which means that the space X is super σ -metrizable. \square

Theorem 2 and [3, 3.1.19] implies an interesting metrization theorem for compact topological spaces.

Corollary 2. *For a compact space X the following conditions are equivalent:*

- (1) *X is metrizable;*
- (2) *X is Hausdorff and σ -metrizable;*
- (3) *X is strongly σ -metrizable;*
- (4) *X is super σ -metrizable.*

The Hausdorff requirement in Corollary 2(2) is essential as shown by the following example.

Example 1. *Let X be a countable infinite space endowed with the Zariski topology*

$$\tau = \{\emptyset\} \cup \{X \setminus F : F \text{ is finite}\}.$$

The space X is compact and σ -metrizable but Hausdorff and not metrizable.

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